Relative extrema and second order conditions

1. $$z = (x-y)^4 + (y-1)^4$$

2. $$z = y + \frac{8}{x} + \frac{x}{y}$$

3. $$z = e^{x-y}\left(x^2 - 2y^2\right)$$

Solution

1. Calculating the first-order conditions:

$$\frac{\partial z}{\partial x} = 4(x-y)^3 = 0$$

$$\frac{\partial z}{\partial y} = -4(x-y)^3 + 4(y-1)^3 = 0$$

From the first equation, we obtain $x = y$ and using this in the second equation:

$$\frac{\partial z}{\partial y} = 4(y-1)^3 = 0$$

$y = 1$, so the critical point is $(1, 1)$. We calculate the Hessian:

$$\frac{\partial^2 z}{\partial x^2} = 12(x-y)^2$$

$$\frac{\partial^2 z}{\partial y^2} = 12(x-y)^2 + 12(y-1)^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -12(x-y)^2$$

$$|H| = \begin{vmatrix} 12(x-y)^2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 + 12(y-1)^2 \end{vmatrix}$$

Replacing in the point, we see that $|H| = 0$, however, since the function is a sum of terms raised to even exponents, we know that the minimum will be 0. Therefore $(1, 1, 0)$ constitutes a minimum.

2. Calculating the first-order conditions:

$$\frac{\partial z}{\partial x} = -8/x^2 + 1/y = 0$$

$$\frac{\partial z}{\partial y} = 1 - x/y^2 = 0$$

From the first equation, we obtain:

$$1/y = 8/x^2$$
$$x^2 = 8y$$

From the second equation, we obtain:

$$1 = x/y^2$$
$$y^2 = x$$

Solving from the first equation:

$$x = \pm\sqrt{8y}$$

And with the second equation:

$$y^2 = \pm\sqrt{8y}$$

The only value that satisfies this is $y = 2$, and with this, we get the value of x:

$$2^2 = x = 4$$

So the critical point is $(4, 2)$. We calculate the second derivatives:

$$\frac{\partial^2 z}{\partial x^2} = 16/x^3$$

$$\frac{\partial^2 z}{\partial y^2} = 2x/y^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -1/y^2$$

Calculating the Hessian:

$$\begin{vmatrix} 16/x^3 & -1/y^2 \\ -1/y^2 & 2x/y^3 \end{vmatrix}$$

Evaluating at the point:

$$\begin{vmatrix} 1/4 & -1/4 \\ -1/4 & 1 \end{vmatrix} = 1/4 - 1/16 = 0.1875 > 0$$

Therefore, since the determinant is positive and $\frac{\partial^2 z}{\partial x^2} > 0$, we have a minimum.

3. Calculating the first-order conditions:

$$\frac{\partial z}{\partial x} = e^{x-y}(x^2 - 2y^2) + 2xe^{x-y} = 0$$

$$\frac{\partial z}{\partial y} = -e^{x-y}(x^2 - 2y^2) - 4ye^{x-y} = 0$$

Dividing everything by e^{x-y}:

$$\frac{\partial z}{\partial x} = (x^2 - 2y^2) + 2x = 0$$

$$\frac{\partial z}{\partial y} = -(x^2 - 2y^2) - 4y = 0$$

From the second equation, we obtain:

$$-x^2 + 2y^2 - 4y = 0$$

Adding to the first equation:

$$2x - 4y = 0$$

From here we get $x = 2y$. Using this in the second equation:

$$-4y^2 + 2y^2 - 4y = -2y^2 - 4y = -2y(y+2) = 0$$

We have $y = 0$ or $y = -2$. This leads to two points: $(0, 0)$ and $(-4, -2)$.

We calculate the second derivatives:

$$\frac{\partial^2 z}{\partial x^2} = e^{x-y}(x^2 - 2y^2) + 2xe^{x-y} + 2e^{x-y} + 2xe^{x-y}$$

$$\frac{\partial^2 z}{\partial y^2} = -e^{x-y}(x^2 - 2y^2) - 2xe^{x-y} - 4e^{x-y} + 4ye^{x-y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -e^{x-y}(x^2 - 2y^2) - 4ye^{x-y} - 2xe^{x-y}$$

Evaluating at $(0, 0)$, the Hessian is:

$$|H| = \begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix} = -8$$

This is a saddle point. Now, evaluating at $(-4, -2)$:

$$|H| = \begin{vmatrix} -0.812 & 1.083 \\ 1.083 & -1.624 \end{vmatrix} = 0.145 > 0$$

And since $\frac{\partial^2 z}{\partial x^2} < 0$, we have a maximum.